Leverage and the Limits of the Possible

How leverage impacts the policy portfolio.

Frederick E. Dopfel

> Soon her eye fell on a little glass box that was lying under the table: she opened it, and found in it a very small cake, on which the words "EAT ME" were beautifully marked in currants. "Well, I'll eat it," said Alice, "and if it makes me grow larger, I can reach the key; and if it makes me grow smaller, I can creep under the door; so either way I'll get into the garden, and I don't care which happens!"
>
> —*Alice in Wonderland*
> Lewis Carroll

If only investors could engineer their target returns to be the size they would like, just as Alice could change her size to reach some goal. Some have suggested we can construct better investment policy portfolios by including bonds leveraged to the risk level of stocks, or more generally by leveraging and deleveraging multiple asset classes. Is there such a magic cake that can improve your portfolio? Or instead, are there *limits of the possible?*

Leverage occurs when borrowed funds are used to boost the investor's capital base for investment. Leverage generates an extra return if there is a positive realized spread between the return on the portfolio and the cost of borrowing; leverage is detrimental if the cost of borrowing exceeds the realized returns. Thus, while leverage may be used to increase the expected return of a portfolio, it also increases the risk proportionally.

Is it possible to improve overall policy portfolio performance with a judicious mix of leveraged and deleveraged asset classes? What are the potential benefits? What are the limits?

I demonstrate the impact of leveraging asset classes by showing examples and reviewing familiar portfolio

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concepts. I first describe an example incorporating only stocks and leveraged bonds under equilibrium assumptions that are consistent with capital asset pricing models typically used for strategic asset allocation purposes to identify a policy portfolio. This exercise demonstrates the basic principles of the way leverage changes the investor's opportunity set under consensus-based assumptions.

I then develop a more detailed case study, based on a hypothetical investor's skill-based assumptions that may reflect an informed view that differs from consensus—effectively, an active approach to asset allocation. The example shows the potential benefits and limitations of leveraging and deleveraging across all asset classes, including the impact of borrowing costs.

LEVERAGE UNDER MARKET EQUILIBRIUM ASSUMPTIONS

The first example includes only stocks, bonds, and the risk-free asset. The simplicity of this framework will help us sort out the pure impact of leverage on the investor's opportunities. A reasonable starting point is to set asset class return expectations consistent with market equilibrium that do not imply any special insight or expertise in forecasting.

Consensus-Based Assumptions

Let us assume that institutional investors, on average, hold 60% in stocks and 40% in bonds, and we may think of this allocation as approximating the market portfolio *M*. There is a clear linkage between the market value of any security, or any portfolio of securities, and expected returns on those same securities. If we assume that the values implicit in the market portfolio are properly priced, then *consensus-based* expected returns for stocks and bonds can be derived that are consistent with investors' consensus 60/40 holdings. This principle for determining consensus-based expected returns is often called *reverse optimization* because the expected return assumptions are the output—rather than an input—of an optimization process (see Sharpe [1974]).

Equilibrium, or consensus-based, assumptions are a good starting point for setting asset class assumptions for strategic asset allocation purposes because they are self-consistent with market values and risks but do not imply any special ability or skill at forecasting expected returns.

Exhibit 1 shows simplified consensus-based asset class assumptions based on this 60/40 market portfolio. The forward-looking risk estimates, and an assumed stock-bond correlation of +0.25, are based on long-run historical experience. Risk, as measured here and in all the other exhibits, is the expected standard deviation of returns. The expected return assumptions are consensus-based, meaning that these assumptions combined with the risk and correlation estimates imply that 60/40 holdings are, in fact, optimal holdings. These assumptions appear reasonable and consistent with long-term historical values, with a 0.75 percentage point additional expected return for bonds over the risk-free rate and a 4.25 percentage point premium for stocks.

But if one expects, say, a higher bond premium and lower stock premium, that would be consistent only if the market portfolio values were closer to 40/60 instead of 60/40, or if the risk estimates were modified. For now, we will stay with the expected return assumptions that are consistent with 60/40 holdings and the risk estimates.

A common statistic used to quantify the risk-adjusted performance of an asset is the familiar Sharpe ratio (SR), defined as: $SR = (r - r_f)/\sigma$. The Sharpe ratio is the expected return in excess of the risk-free rate divided by the standard deviation of return. The market portfolio *M* has the highest expected Sharpe ratio, 0.272 in this example.

Note that the SR for stocks is higher than the SR for bonds. This result simply falls out of our risk and correlation assumptions and the assumption that bonds and stocks are properly priced in the aggregate. Sharpe ratios for various asset classes ought to differ from each other—

EXHIBIT 1
Consensus-Based Asset Class Assumptions

	Risk free	Bonds	Stocks	Portfolio M
Expected return	4.25%	5.00%	8.50%	7.10%
Risk (standard deviation)	0.00%	6.00%	16.00%	10.46%
Sharpe ratio	–	0.125	0.266	0.272
Treynor ratio	–	0.029	0.029	0.029

and they do—because securities are *not* priced according to Sharpe ratios.

The Treynor ratio (TR), named after Jack Treynor, co-creator of the capital asset pricing model (CAPM) with Sharpe and Lintner, is shown in the last row of the table, defined as: $TR = (r - r_f)/\beta$. The Treynor ratio measures the expected return in excess of the risk-free rate divided by the beta of the asset class (with respect to the market portfolio). The numerator is the same as the Sharpe ratio but the denominator is a different measure of risk.[1]

As we know, beta incorporates the covariance of an asset class with the market portfolio, so it evaluates the risk of the asset class in an overall portfolio context. Using beta and the TR is consistent with the CAPM and is a better measure for understanding the relative pricing of asset classes or securities in general. This explains why in Exhibit 1 the SR varies, but the TR is equal for the two asset classes—it signifies consistent pricing of market risk that is achieved with consensus-based assumptions.

The Sharpe ratio of the market portfolio specifies the capital market line (CML), a linear relationship between expected return and risk (standard deviation) of portfolios composed of the market portfolio and the risk-free rate. The Treynor ratio specifies the security market line (SML), a linear relationship between expected return and systematic risk (beta). According to the CAPM, all asset classes lie on the SML, but they do not in general lie along the CML because the correlations of asset classes with the market portfolio vary.

This is equivalent to my earlier statements that, under the assumptions of the CAPM, Treynor ratios of asset classes are constant, while Sharpe ratios are not. These principles may not hold perfectly in practice, but they are a useful point of departure in assessing equilibrium asset class assumptions as typically used for policy portfolios (strategic asset allocation).

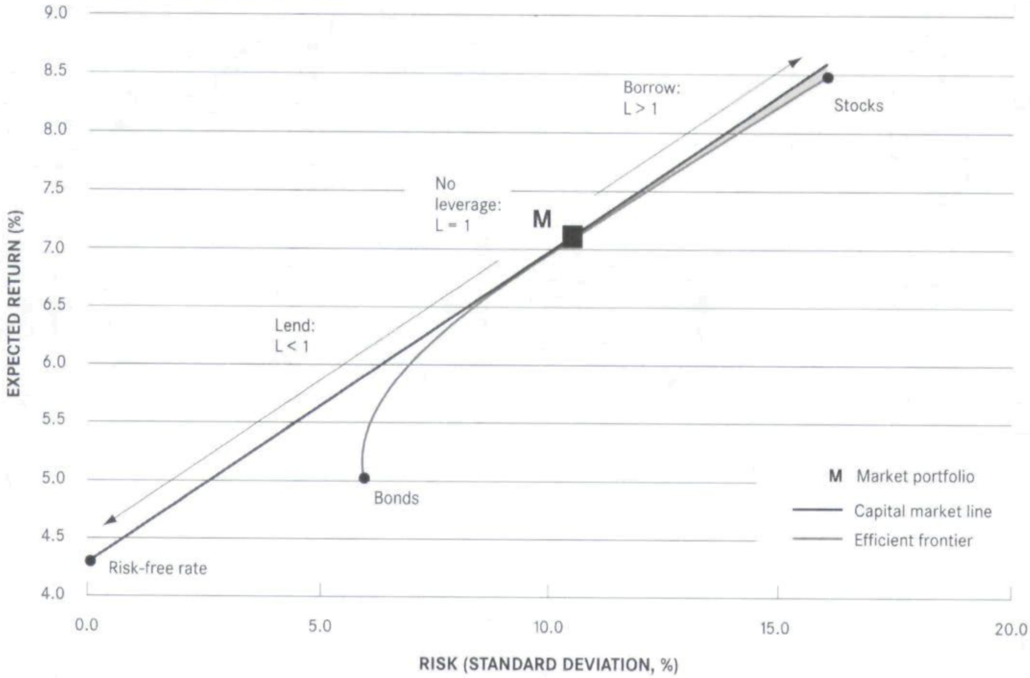
Efficient Frontiers and the Capital Market Line

Exhibit 2 shows the efficient frontier, the mean-variance efficient portfolios for the consensus-based assumptions in Exhibit 1. The capital market line is drawn from the risk-free rate through the market portfolio *M*; in the case of consensus-based assumptions, the CML is tangent to the efficient frontier at the market portfolio *M*, which reflects the fact that *M* is a super-efficient portfolio—the highest Sharpe ratio portfolio.

The capital market line is a sort of super-efficient frontier, allowing for leverage. It is constructed by leveraging the market portfolio—all asset classes are leveraged up or down in the same proportion. It would be the

EXHIBIT 2

Efficient Frontier and Capital Market Line



frontier that a standard optimizer would automatically calculate if it could go either long or short cash.

Points on the CML to the left of M (left-CML) correspond to holding less than 100% (leverage factor $L < 1$) in the market portfolio and lending the remainder at the risk-free rate. Point M on the CML is, by definition, a fully invested 100% market portfolio; that is, no leverage ($L = 1$). Points on the CML to the right of M (right-CML) correspond to borrowing at the risk-free rate to invest more than 100% ($L > 1$) in the market portfolio.

The left-CML is often considered part of the traditional efficient frontier because lending at the risk-free rate is always possible, although rarely acted on. The right-CML represents a leveraging of the market portfolio, but is usually neglected for practical asset allocation purposes because of prohibitions against leverage or the additional costs of borrowing.

The potential benefit of this type of leverage is measured by the gap in expected return between the right side of the CML and the ordinary efficient frontier. We can see here that the benefit, even before the costs of borrowing, appears very slight except at very high levels of risk.

Incorporating Leveraged Bonds

What about separately leveraging asset classes, rather than leveraging all asset classes in the same proportion? For simplicity, in this case we leverage the returns of only the bond portfolio by borrowing funds to invest in a larger portfolio of bonds than our initial capital would otherwise support.[2]

Suppose that for every initial dollar of investment we borrow an additional ($L - 1$) dollars at the risk-free rate and then invest the entire L dollars in bonds. For example, if the leverage factor is $L = 1.5\times$, we are borrowing $0.50 = $1.50 − $1.00 at the risk-free rate to add to our initial $1.00, thereby investing a net $1.50 in bonds. Conversely, leverage includes the idea of deleverage where the leverage factor is less than one, say, $L = 0.75\times$. Then we are lending $0.25 = $0.75 − $1.00 from our initial $1.00, thereby investing a net $0.75 in bonds.

In general, if the leverage factor is L, the expected return for the bond investment is:

$$r_b(L) = r_f + L(r_b - r_f)$$

and the expected risk is:

$$\sigma_b(L) = L\sigma_b$$

The expected return of the bond portfolio is increased by leverage ($L > 1$) only if the expected return of unleveraged bonds is higher than the risk-free rate. In any case, the Sharpe ratio for leveraged bonds is:

$$\begin{aligned} SR_b(L) &= \frac{r_f + L(r_b - r_f) - r_f}{L\sigma_b} \\ &= \frac{r_b - r_f}{\sigma_b} \equiv SR_b \end{aligned} \tag{1}$$

Equation (1) shows an important result that we want to highlight: *The Sharpe ratio is unaffected by leverage* if borrowing (lending) occurs at the risk-free rate, a result that holds for any asset class. It certainly makes sense that the risk-adjusted return, embodied by the SR, should be unaffected by leverage. As expected, *the Treynor ratio is also unaffected by leverage* because the numerator (excess return) and the denominator (beta) change in the same proportion with leverage:

$$\begin{aligned} TR_b(L) &= \frac{r_f + L(r_b - r_f) - r_f}{L\beta} \\ &= \frac{r_b - r_f}{\beta} \equiv TR_b \end{aligned} \tag{2}$$

Exhibit 3 shows the impact of leveraging the bonds at $L = 1.5\times$ leverage and at $L = 2.7\times$ leverage. One times

EXHIBIT 3
Impact of Leverage on Bond Returns

		Bonds with leverage			
	Risk-free rate	1X	1.5X	2.7X	Stocks
Expected return	4.25%	5.00%	5.38%	6.25%	8.50%
Risk (standard deviation)	0.00%	6.00%	9.00%	16.00%	16.00%
Sharpe ratio	–	0.125	0.125	0.125	0.266
Treynor ratio	–	0.029	0.029	0.029	0.029

leverage ($L = 1$) means the bonds are unleveraged. Note that the higher leverage of 2.7× corresponds to equating the risk of leveraged bonds to the risk of equities at 16.0% as shown in the exhibit.

Exhibit 3 also confirms that the Sharpe ratio and the Treynor ratio are unchanged by leverage. Next we shall see if there are benefits to replacing unleveraged bonds with leveraged bond assets when compared with the ordinary efficient frontier (calculated without asset class leverage) and the capital market line.

Exhibit 4 repeats the original efficient frontier and capital market line from Exhibit 2, but includes new frontiers combining stocks with bonds at 1.5× and 2.7× leverage. The efficient frontiers with leveraged bonds are each tangent to the CML but at progressively higher risk levels, reflecting the higher-risk leveraged bonds in the mix. As we change the leverage of the bond asset class, and connect the dots between the points of tangency, the CML is traced out for different levels of risk and leverage. The CML is like a new frontier that incorporates portfolios for all combinations of leveraged and deleveraged bonds.

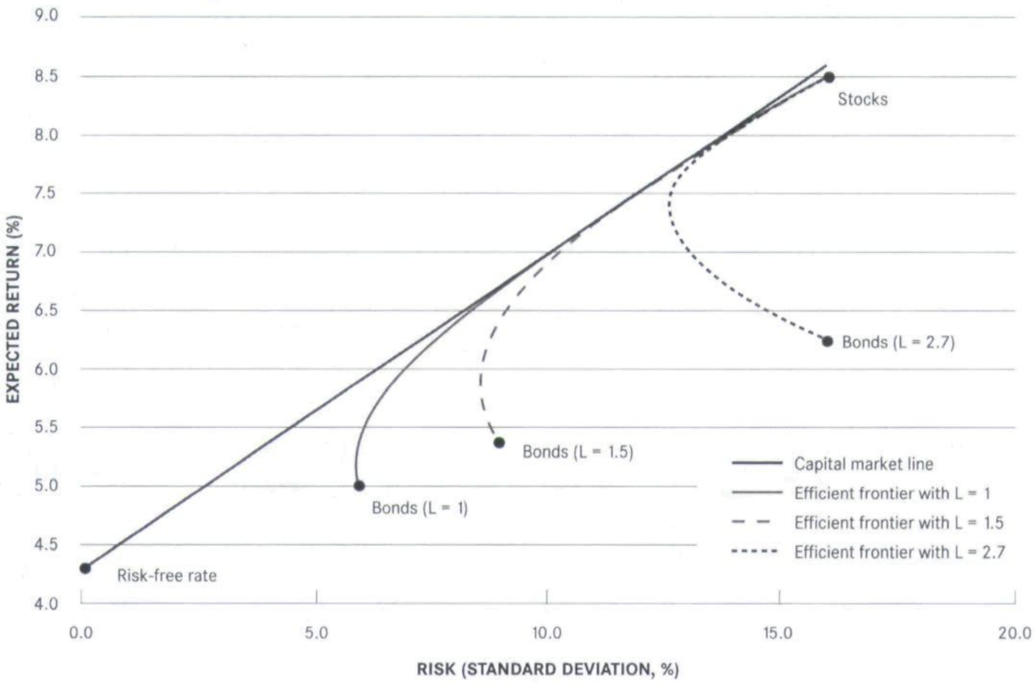
The differences between the new frontiers and the old frontier shown in Exhibit 4 are negligible for a wide range of risk budgets. While the frontiers that use leveraged bonds show higher expected returns within a band of higher risk levels, *all* efficient frontiers lie at or below the capital market line. This result is not surprising, as there has been no change in Sharpe ratios for any of the asset classes. Again, this is a highly stylized example with only stocks and leveraged bonds, but the principle that any frontier with one or more leveraged asset classes will always lie at or below the associated CML also holds for multiple asset classes under non-equilibrium assumptions.[3]

Benefits of Leverage?

We see in Exhibit 4 that there are no additional benefits of *separately* leveraging asset classes. The capital market line, leveraging portfolio M alone, is preferred to all portfolios on the ordinary efficient frontier (i.e., equal to or above), but it is also preferred to (equal to or above) *any* frontier using a leveraged bond asset class. In this example, the market portfolio M, which defines the capital market line, is the highest Sharpe ratio portfolio, and thus the CML represents the *limits of the possible.*

Noting that the capital market line is the limit of the possible, we observe in Exhibits 2 and 4 that the actual difference between the CML and the ordinary efficient frontier is negligible. Consensus-based assumptions imply that the capital market line is tangent to the efficient frontier in the neighborhood of typical risk budgets (10%–11%, like portfolio M). Within that range of risk budgets,

EXHIBIT 4
Efficient Frontiers with Leveraged Bonds



the frontier is fairly straight with a slope comparable to the CML. Thus, there is only a slim gap between the most efficient portfolios without leverage and the CML at typical risk budgets. Leverage of the market portfolio (up or down on the CML) is beneficial only if the investor has a high enough risk budget to make a material gap between the CML and the ordinary efficient frontier. We can see from Exhibits 2 and 4 that this would have to be an uncommonly high risk budget.

LEVERAGE UNDER NON-EQUILIBRIUM ASSUMPTIONS

The discussion so far has focused on a highly simplified example with consensus-based assumptions. In practice, real-world investors often use asset class assumptions that are *not* consensus-based and that imply that one or more asset classes are materially mispriced for some reason. These investors may have an informed view that is different from consensus expectations. Whether these investors can use leverage to improve investment policy, conditional on their specific forecasts, is a separate question.

In the simplified example, under the economic restrictions of equilibrium (consensus-based) assumptions, leverage affords negligible benefits to investment policy performance except at unacceptably high risk levels for most institutional investors. Now we can consider the impact of using leverage with non-equilibrium (skill-based) assumptions that represent a hypothetical investor's informed view of expected asset class returns.

Using non-equilibrium assumptions means making forecasts of the returns, risks, or correlations of the asset classes that are different from consensus. Taking bets by changing beta (systematic exposures) depending on these informed views amounts to making bets on alpha (active exposures), an active management discipline sometimes called tactical asset allocation. Because this is inherently an active decision, some caution is in order—active management should be pursued only if the investor has above-average skill in making these forecasts.

Investor Assumptions and the Conditional Efficient Frontier

The investor assumptions portrayed in Exhibit 5 include a *conditional* efficient frontier showing a set of mean-variance efficient portfolios (excluding cash). I use the term, 'conditional' because the performance of these portfolios will be conditional on the specific investor's skill-based forecasts and expectations, rather than on consensus expectations—effectively *active* asset allocation. In the example, I follow the usual practice of taking risk (standard deviation) and correlation assumptions from an analysis of historical returns, although some active asset allocation work could also be done by making specific forecasts on these terms.

Overall, the investor has assigned higher expected returns to the higher-risk (higher standard deviation) asset classes. But only by examining these assumptions in more detail can we determine that the expected returns are not consensus-based assumptions, nor are they governed by the capital asset pricing model or by any of its close cousins.

While this fact is not readily apparent in Exhibit 5, Exhibit 6 shows that the Treynor ratios for the asset classes vary significantly, implying that the implicit prices of risk for the various asset classes are not equated as in the case of consensus-based assumptions. In other words, the asset classes—if plotted in expected return and beta space—would not follow a neat security market line, as is the case in the earlier example.

It is also noteworthy that the assumptions do not appear at all random—this investor is a bond bull, with the implied Treynor ratios for the bond asset classes consistently higher than the Treynor ratios for all stock-related asset classes. Thus, in expected return and beta space, the bond-like asset classes would plot high, and the equity-like asset classes would plot low.

Simply put, this investor believes that bonds are underpriced and stocks are overpriced, according to some special insights the investor has into the market that justify differences from consensus expectations. Further, we assume that the investor is confident in these views and wishes to make asset allocation decisions in combination with leverage based on these insights.

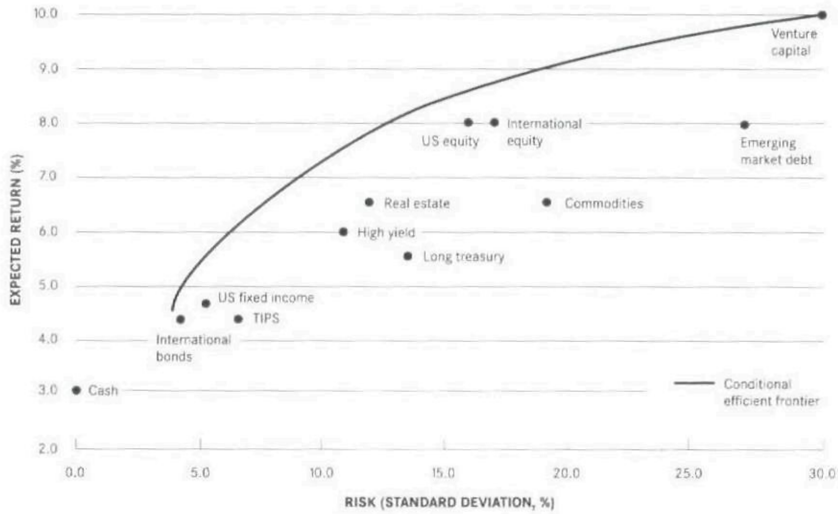
Super-Efficient Portfolio and the Super CML

Exhibit 7 repeats the same conditional efficient frontier as in Exhibit 5 (without showing the individual asset classes), but defines two portfolios of special interest and the *super capital market line*. Portfolio Q represents the *consensus portfolio* (a proxy for the market portfolio M), and portfolio S represents the *super-efficient portfolio* based on the investor's skill-based insights.

Portfolio Q is called the consensus portfolio because it is calculated as the current holdings of the 200 largest

EXHIBIT 5

Investor's Asset Class Assumptions and Conditional Efficient Frontier



Asset class	Expected return (%)	Standard deviation (%)
Cash	3.00	0.00
International bonds	4.25	4.50
US fixed income	4.50	5.50
TIPS	4.25	6.75
High yield	6.00	11.00
Long treasury	5.50	13.50
Real estate	6.50	12.00
International equity	8.00	16.50
US equity	8.00	15.50
Commodities	6.50	19.00
Emerging market debt	8.00	27.00
Venture capital	10.00	30.00

Asset class	Int'l bonds	US fixed income	TIPS	High yield	Long treasury	Real estate	Int'l equity	US equity	Commod-ities	EM debt	Venture capital
International bonds	1.00	0.70	0.50	0.10	0.60	0.35	0.15	0.10	0.10	0.10	0.10
US fixed income	0.70	1.00	0.70	0.20	0.80	0.45	0.10	0.20	0.10	0.10	0.20
TIPS	0.50	0.70	1.00	0.15	0.60	0.30	0.05	0.15	0.25	0.05	0.15
High yield	0.10	0.20	0.15	1.00	0.10	0.30	0.40	0.50	0.10	0.45	0.60
Long treasury	0.60	0.80	0.60	0.10	1.00	0.35	0.10	0.10	0.10	0.10	0.10
Real estate	0.35	0.45	0.30	0.30	0.35	1.00	0.40	0.60	0.10	0.25	0.50
International equity	0.15	0.10	0.05	0.40	0.10	0.40	1.00	0.70	0.10	0.15	0.45
US equity	0.10	0.20	0.15	0.50	0.10	0.60	0.70	1.00	0.10	0.10	0.70
Commodities	0.10	0.10	0.25	0.10	0.10	0.10	0.10	0.10	1.00	0.10	0.10
Emerging market debt	0.10	0.10	0.05	0.45	0.10	0.25	0.15	0.10	0.10	1.00	0.10
Venture capital	0.10	0.20	0.15	0.60	0.10	0.50	0.45	0.70	0.10	0.10	1.00

EXHIBIT 6
Treynor Ratios Implied by Investor's Assumptions

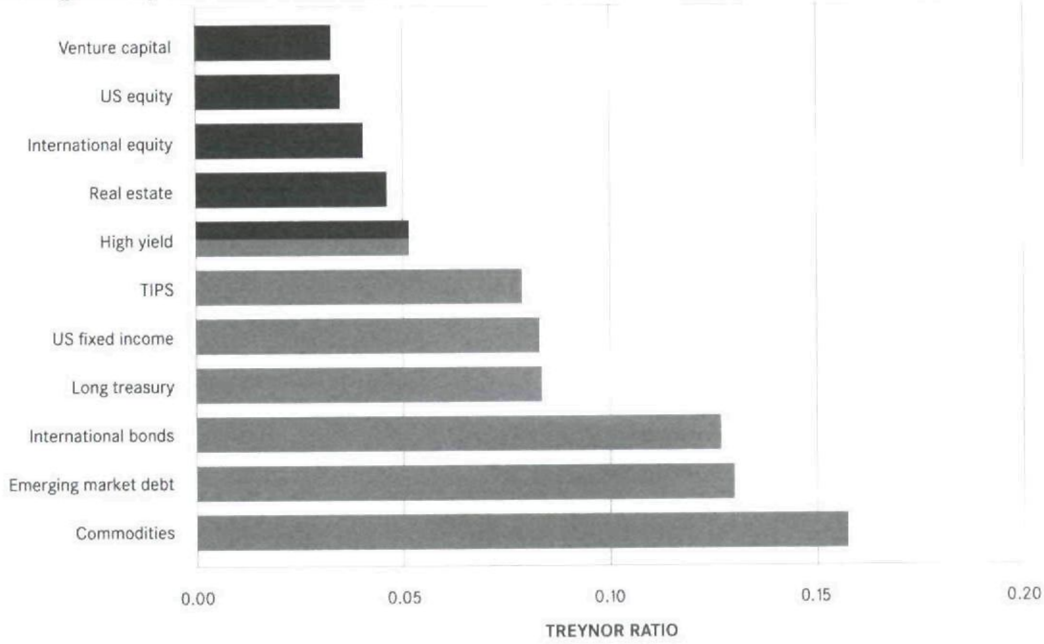
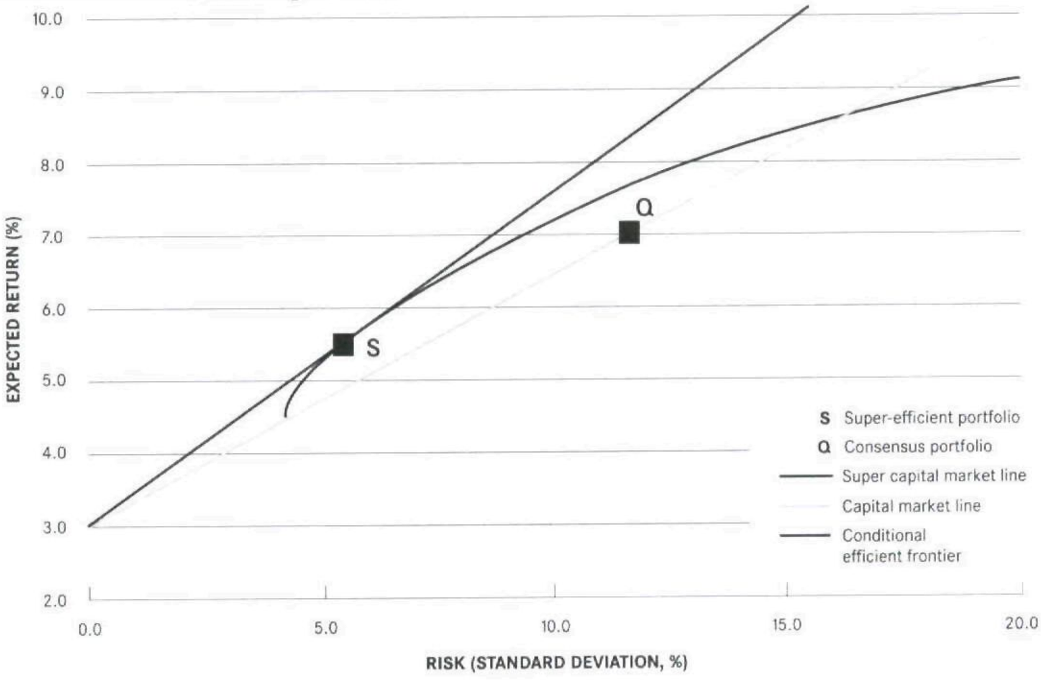


EXHIBIT 7
Conditional Efficient Frontier and Super CML



U.S. defined-benefit pension plans (although there are alternative proxies for the market, each with good supporting arguments). To incorporate portfolio Q into Exhibit 7, we use the investor's conditional return expectations and the inputs described above.

In the earlier example, portfolio M is directly on the efficient frontier—because the frontier itself is derived with expected returns chosen to be consistent with market holdings—and it is the highest Sharpe ratio portfolio. But under our investor's (conditional) expected return assumptions, portfolio Q does *not* lie on the conditional efficient frontier, as shown in the exhibit, because our investor's expectations do not coincide with consensus expectations. Clearly, portfolio Q is not a high Sharpe ratio portfolio either—in fact, it has no special significance to this investor, as many other portfolios appear to be superior in mean-variance terms.

A line beginning at the risk-free rate and passing through portfolio Q represents a new capital market line (CML). It is analogous to our first ordinary capital market line, showing portfolios that leverage the market portfolio up and down, except that the actual placement of the CML depends, of course, on the placement of portfolio Q. Unlike the market equilibrium case, the portfolios on the CML appear grossly inefficient in the eyes of this investor because of some presumed unique insights.

Portfolio S is called the super-efficient portfolio because it is the highest Sharpe ratio portfolio on the basis of the investor's assumptions (active insights). In our earlier discussion using consensus expectations, the market portfolio and the highest Sharpe ratio portfolio were one and the same. But in this instance, portfolio S has lower risk than portfolio Q, with a much higher allocation to bonds than average institutional portfolios (79%) and a much lower allocation to stocks (21%). In effect, the investor believes that holdings in these proportions are the ideal mix for the current market conditions, even though they differ markedly from consensus holdings.

The super-capital market line (super CML) is a line drawn from the risk-free rate through the super-efficient portfolio S and is tangent to the conditional efficient frontier at that point. The super CML reflects the opportunities for leverage—lending (to the left of portfolio S) and borrowing (to the right of portfolio S) in combination with investment in portfolio S.

Remembering that the Sharpe ratio is unaffected by leverage, it is not possible to construct a portfolio with a higher Sharpe ratio by leveraging and deleveraging the asset classes separately. The complete opportunities presented by leverage are realized by leveraging the super-efficient portfolio S with all assets leveraged up or down by the same proportions as in that portfolio.

Calculation of the conditional efficient frontier and the super CML and the overall conclusions about the possibilities and limitations of leverage are also predicated on our covariance assumptions. Investors might have conditional expectations for correlation that differ from equilibrium estimates that would affect the outcomes. But as with all active disciplines, active correlation estimates are only as good as the skill level of the investor making the forecasts.

For example, realized correlations may appear unstable over short periods, and some investors may be tempted to ignore them or to simplify by asserting that the expected asset correlations are zero, thereby greatly reducing expected risk. It is then theoretically possible to construct a portfolio with a very high expected return of 10% or more with expected risk in the range of a traditional portfolio. This extraordinary result does not disprove our theory, because that portfolio lies along a new (theoretical) super CML based on a new (theoretical) super-efficient portfolio that ignores correlations between asset classes. But asset class correlations should *not* be ignored even if they are hard to forecast (see Dopfel [2003]).

The super CML is the limit of the possible in the eyes of our hypothetical investor just as the ordinary CML is the limit of the possible for the investor using equilibrium assumptions. To the latter and more conservative investor who is not prepared to claim market timing skill but is prepared to challenge asset class assumptions that deviate from consensus, the super CML is *beyond* the limits of the possible.

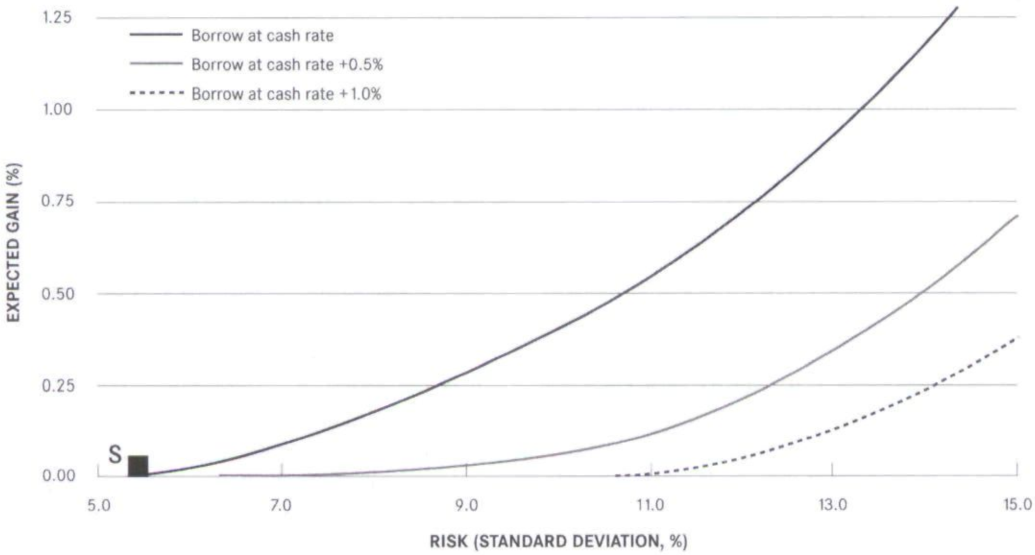
Benefits of Leverage with Conditional Expectations

The maximum benefit of leverage, conditional on the investor's expectations, may be quantified by measuring the expected return difference (or gap) between the super-capital market line and the conditional efficient frontier. We should also account for the fact that there may be a higher cost of borrowing than the cash rate.

Exhibit 8 plots the potential benefit in terms of expected return improvement for various risk budgets and borrowing rates. Portfolio S lies right on the conditional EF at its tangency with the super CML, so there is no benefit of leverage at that policy risk budget. The potential benefit widens and becomes material at higher

EXHIBIT 8

Potential Gain from Leverage Depending on Risk Budget and Borrowing Rates



risk levels, although the benefit is diminished if the cost of borrowing is assumed to exceed the cash rate.

For example, at typical policy risk budgets of around 11.5%, the potential gain in expected return is approximately 0.60%. But this hypothetical gain is reduced to less than 0.20% at borrowing rates of cash plus 0.50% (3.50%), reduced to less than 0.05% at borrowing rates of cash plus 1.0% (4.0%), and disappears at higher borrowing costs. If we were to acknowledge that, in reality, borrowing costs are volatile rather than risk-free, the potential benefit of leverage would be further limited for this investor.

We have generously assumed that the investor in this example has a skillful, informed view of asset class expected returns. Expected return assumptions should not be set casually, but we may ask, hypothetically, what sort of assumptions would generate the highest potential gain from leverage?

Thinking about the geometry of Exhibit 7, the potential gains to leverage are highest if the expected return spread between bonds and the risk-free rate is assumed to be as wide as possible, and if the expected return spread between stocks and bonds is assumed to be as narrow as possible. Under these conditions, the new super CML would be steep with a tangency at a lower-than-average risk level, and the new conditional efficient frontier would be fairly flat at typical risk levels. These geometric relationships would yield the widest gap between the super CML and the conditional efficient frontier at typical risk levels.

Even so, the potential gains from leverage depend on both skillfully defined assumptions that are different from consensus expectations and the ability to borrow at relatively low rates.

CONCLUSION

Leveraging the bond asset class—or any other asset class—lets us construct new policy portfolios that may outperform or underperform unleveraged portfolios. None of these policy portfolios, however, can be superior (in a mean-variance sense) to portfolios represented by the super-capital market line, which simply leverages up or down a super-efficient portfolio, the highest Sharpe ratio portfolio. Any portfolio incorporating leveraged and deleveraged asset classes lies at or below this boundary. In other words, the super CML represents the *limits of the possible* regardless of asset class assumptions.

In an equilibrium context, the market portfolio is the highest Sharpe ratio portfolio; the super CML and the traditional capital market line are one and the same. The CML boundary is very constraining in practice because most investors set risk budgets that are not so far—left or right—from the risk budget of the consensus portfolio. Therefore, under equilibrium (consensus-based) assumptions, the potential benefit of any kind of leverage from a strategic policy viewpoint is very limited.

Moving beyond equilibrium, many investors prefer to use asset class assumptions that reflect perceived mispricing of various asset classes, as in our example under skill-based (non-equilibrium) assumptions. Again, the opportunity set with leverage is defined by the super-capital market line, which leverages the super-efficient portfolio calculated from the investor's conditional expectations. But the composition of the super-efficient portfolio may differ dramatically from typical strategic allocations, depending on the difference between the investor's views on expected returns and the consensus.

The benefit of judiciously applied leverage, though, is still limited by the extent of the gap between the super CML and the conditional efficient frontier and by additional borrowing costs. It is not surprising, then, that the use of leverage in the policy portfolio, while an intriguing idea, is very rarely seen. Leverage is more commonly found in higher-risk active portfolios.

Active managers sometimes use leverage—combined with a number of proprietary insights—to optimize expected alpha, but the possible benefit of leverage in active strategies occurs only in the presence of skillful insights. In the policy context, there is no automatic value-added from simple leverage either, and realizing benefits from leverage requires, at a minimum, the investor to express an informed and confident view that differs from consensus.

Finally, one should be wary of any claims that leverage is a magic cake that enables investors to outperform traditional investment policies in all environments. Leverage is not without its risks, as Alice learned the hard way:

> "That's quite enough—I hope I shan't grow any more—As it is, I can't get out at the door—I do wish I hadn't [eaten] quite so much!" Alas! It was too late to wish that! . . . Still she went on growing, and, as a last resource, she put one arm out of the window, and one foot up the chimney, and said to herself "Now I can do no more, whatever happens. What WILL become of me? . . . Why, there's hardly room for [me], and no room at all for any lesson-books."
>
> —*Alice in Wonderland*
>
> Lewis Carroll

ENDNOTES

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[1]See French [2003] for an interesting discussion of the development of the single-period discrete-time capital asset pricing model, including references and comparisons of Treynor [1961], Treynor [1962], and Sharpe [1964]. Tobin [1958] is credited with adding the notion of leverage to portfolio theory by including the risk-free asset and deriving a *linear opportunity locus*.

[2]This traditional definition of leverage is very narrow, as leverage includes shorting assets other than cash. Effectively, that is what is going on in relative terms as we underleverage ($L < 1$) various asset classes in order to fund the overleveraging ($L > 1$) of other asset classes.

[3]The strategic asset allocation problem is to maximize expected return subject to a portfolio expected risk budget, where r is the assumed vector of expected excess returns and V is the assumed covariance matrix. This problem can be expressed as the maximization over holdings, represented by vector h, of the objective: $U = r'h - \lambda h'Vh$ where λ is a risk penalty parameter (scalar). The first-order conditions imply: $r - 2\lambda Vh = 0$ and then optimal holdings: $h = (1/2\lambda)V^{-1}r$.

A reasonable economic assumption is that the optimal holding of each asset is positive, so that h is a positive vector. The expected return of an optimal portfolio is $r'h = (1/2\lambda)r'V^{-1}r$, and its variance is $h'Vh = (1/2\lambda)^2 r'V^{-1}r$. By substitution, the maximum Sharpe ratio for assumptions $[r, V]$ is:

$$SR_{\max} \equiv \frac{r'h}{\left(h'Vh\right)^{1/2}}$$

$$= \frac{(1/2\lambda)r'V^{-1}r}{\left((1/2\lambda)^2 r'V^{-1}r\right)^{1/2}} = \left(r'V^{-1}r\right)^{1/2}$$

The maximum Sharpe ratio defines the super-capital market line. Here, the super CML refers to both the ordinary CML in the case of equilibrium assumptions and the super CML in the general case. The super CML is equal to the set $\{(\sigma, r): r = SR_{\max}\sigma\}$. All portfolios must lie at or below the super CML in (σ, r) space. The intent is to show that the super CML is unchanged by the introduction of leverage. Equivalently, it is sufficient to show that the maximum Sharpe ratio is unchanged by leverage.

Assume that any individual asset class j can be leveraged L_j so that: $r_j(L) = L_j r_j$, read as the leveraged expected excess return is equal to the leverage factor multiplied by the unleveraged expected excess return. The covariance matrix is also rescaled with $\sigma_{ij}(L) = L_i L_j \sigma_{ij}$, read as the leveraged covariance is equal to the product of the leverage factors multiplied by the unleveraged covariance.

We define the diagonal matrix L where the i-th diagonal element is L_i. Then, the vector of leveraged expected excess returns is $r(L) = Lr$, and the leveraged covariance matrix is $V(L) = LVL$, so the optimization problem for assumptions $[r, V]$ can be replaced with assumptions $[Lr, LVL]$ to incorporate leverage L. The first-order conditions imply:

$$h(L) = (1/2\lambda_L)L^{-1}V^{-1}r = (\lambda/\lambda_L)L^{-1}h$$

or:

$$\lambda_L L h(L) = \lambda h$$

where $h(L)$ is the vector of optimal holdings with leverage L.

Thus the optimal holdings after incorporating any choice of leverage are the original holdings just scaled by the leverage factors. The net effect is that the investor will hold the same underlying portfolio as without leverage. The only difference is a proportional scaling of risk and return according to the investor's risk preferences.

The resulting maximum Sharpe ratio with leverage is:

$$SR_{\max}(L) = \left[(Lr)'(LVL)^{-1}(Lr)\right]^{1/2} = \left(r'V^{-1}r\right)^{1/2} = SR_{\max}$$

Therefore, the maximum Sharpe ratio is unaffected by incorporating leverage. As the maximum Sharpe ratio defines the super CML, the upper boundary of the investor's opportunities is also unaffected by leverage.

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